X-550-68-156

PREPRINT

NASA TM X- 70546

GODDARD ORBIT REFERENCE SYSTEMS

JOSEPH W. SIRY

FEB 74
RECEIVED
NASA STI FACILITY
INPUT BRANCH

FEBRUARY 1968

GSFC — GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND

(NASA-TM-X-70546) GODDARD ORBIT REFERENCE SYSTEMS (NASA) 42 p

N74-71017

Unclas
00/99 27028

X-550-68-156

PREPRINT

GODDARD ORBIT REFERENCE SYSTEMS

Joseph W. Siry

Mission and Trajectory Analysis Division

February, 1968

GODDARD SPACE FLIGHT CENTER

Greenbelt, Maryland

CONTENTS

		Page
I.	INTRODUCTION	1
II.	REFERENCE SYSTEMS AND CONVENTIONS	1
	A. Coordinate Systems	1
	B. Vectors and Transformations	5
	C. Time Systems	13
III.	COORDINATE TRANSFORMATIONS	15
	A. Fundamental Reference Systems	15
	B. Earth-fixed Coordinate Systems	21
	C. An Ecliptic Reference System	25
	D. Lunar Coordinate Systems	26
IV.	APPLICATIONS	31
	A. Vectors Associated With the Earth	31
	B. Vectors Associated With the Moon	33
	C. Combination and Representation of Vectors and Transformations	38
V.	REFERENCES	39

I. INTRODUCTION

This discussion is concerned with reference systems associated with the Goddard General Orbit Determination System and its use. The first section deals in a general way with conventions, definitions, terminology and notation. Additional details relating to these topics are contained in following sections.

Many of the conventions employed concerning definitions, terminology and notation follow traditional usage as employed for example in Reference 1. Those which pertain to the present material are reviewed briefly. Standard notational practice is supplemented occasionally in order to facilitate the discussion.

II. REFERENCE SYSTEMS AND CONVENTIONS

A. Coordinate Systems

Coordinate systems associated with the Goddard General Orbit Determination System are referred to the fundamental coordinate systems which are based upon the mean equator and equinox of the earth. A typical coordinate system of this type is defined in the following way. The origin of the coordinate system is defined to be at the center of mass of the earth. The fundamental plane of the coordinate system is the earth's mean equatorial plane associated with a particular epoch which will be denoted by means of a symbol such as t, or t_c, t_1, t_2, $\cdots$. The fundamental direction in this plane is that of the mean equinox associated with this epoch. The coordinate system is inertially oriented, i.e., its reference directions are fixed with respect to the "fixed" stars. The coordinate system is a right-handed, orthogonal one. It can be conveniently specified in terms of a triple of unit vectors. The symbols $\underline{i}(t)$, $\underline{j}(t)$, and $\underline{k}(t)$, are frequently used for this purpose. Traditionally, the vector $\underline{i}(t)$, is in the direction of the vernal mean equinox, the vector $\underline{k}(t)$, is in the direction of the earth's north polar axis, and the vector $\underline{j}(t)$, is chosen so as to complete the right-handed, orthogonal set. For convenience, in the following discussion, the symbols $\underline{u}_i(t)$, where $i = 1, 2, 3$, will be used in lieu of $\underline{i}(t)$, $\underline{j}(t)$, and $\underline{k}(t)$, respectively. When the range of the index is not indicated in what follows, it will be understood to be the same as in this case.

In order to facilitate the discussion, the following notational system will be employed to indicate the principal characteristics of the coordinate system.

The origin will be indicated by means of a prefixed subscript. For example,

$$ {}_{\oplus}\underline{u}_i(t) \ , \quad \text{and} \quad {}_{☾}\underline{u}_i(t) \ , \tag{2-1} $$

will denote earth-centered and lunar-centered coordinate systems, respectively. The fundamental plane and direction will be indicated by means of symbols following the index. The first symbol will denote the entity with which we associate the fundamental plane. Typical choices for this symbol are $\oplus$ and ☾. The next symbol or symbols will denote the character of the fundamental plane and direction, e.g. M for mean, T for true, etc. Thus, for example

$$ {}_{\oplus}\underline{u}_{i\,\oplus M}(t) \ , \quad \text{and} \quad {}_{\oplus}\underline{u}_{i\,\oplus T}(t) \ , \tag{2-2} $$

will denote earth-centered coordinate systems based upon the earth's mean equator and equinox, and true equator and equinox, respectively.

If the fundamental plane and the fundamental direction have different characters, two symbols will be used. The first will denote the fundamental plane, and the second will denote the fundamental direction. Thus,

$$ {}_{\oplus}\underline{u}_{i\,\oplus TS}(t) \ , \tag{2-3} $$

will denote an earth-centered coordinate system based upon the earth's true equator and a reference direction which will be referred to as the space equinox. The precise nature of this coordinate system will be defined later in the discussion.

When the fundamental plane is the ecliptic, the symbol, C, will appear immediately following the index. The next symbol will denote the entity with which we associate the fundamental direction, and the symbol following it will denote the character of that fundamental direction. Thus

$$ {}_{\oplus}\underline{u}_{i\,C\oplus M}(t) \ , \tag{2-4} $$

denotes an earth-centered coordinate system based upon the ecliptic and the earth's mean equinox, which is defined as the ascending node of the ecliptic on the mean equator of the earth.

In the case of another entity, the fundamental direction will normally be defined in analogous fashion, i.e., as the ascending node of the ecliptic on the specified equator of the object of interest. The notation will also be analogous.

For example,

$$ {}_{\leftmoon}\underline{u}_{i\,C\leftmoon T}(t) , \tag{2-5}$$

denotes a lunar-centered coordinate system based upon the ecliptic and the ascending node of the ecliptic on the true equator of the moon.

It is helpful, in certain cases, to employ, also, an inertially oriented coordinate system whose axes coincide, respectively, at the epoch, t, with those of a set which are associated directly with a body such as the earth. Thus, for example, the symbols

$$ {}_{\oplus}\underline{u}_{i\,\oplus TG}(t) , \tag{2-6}$$

will denote an earth-centered coordinate system for which the reference plane is the earth's true equator and the reference direction is on the Greenwich meridian plane.

It is convenient, at times, to employ coordinate systems which are not inertially oriented, but are rotating. Generally, in the following discussion, sensible rotations can be thought of as occurring about an axis which is normal to the fundamental plane of an appropriately chosen inertially oriented coordinate system. In such cases, the fact that, relative to an inertially oriented coordinate system, the coordinate system of interest is rotating about the axis normal to its fundamental plane will be indicated by enclosing the symbol for its reference direction in parentheses. Thus,

$$ {}_{\oplus}\underline{u}_{i\,\oplus T(G)}(t) , \tag{2-7}$$

denotes an earth-centered coordinate system which is rotating at the earth's instantaneous rate of rotation about an axis normal to its fundamental plane, which is the earth's true equatorial plane, and whose reference direction is that of Greenwich.

Coordinate systems employed in following discussion will be inertially oriented unless the contrary is indicated.

In order to simplify the notation, when the meaning is clear, symbols denoting the earth will be understood but not written. For example, the symbols

$$\underline{u}_{iT}(t) \ , \quad \text{and} \quad {}_{\oplus}\underline{u}_{i\oplus T}(t) \ , \tag{2-8}$$

will both denote the same coordinate system.

B. Vectors and Transformations

The components of a vector, $\underline{v}(t)$, referred to a coordinate system, $\underline{u}_i(t)$, can be specified by means of either the row matrix,

$$\left\| \underline{v}_j(t) \cdot \underline{u}_i(t) \right\| \ , \tag{2-9}$$

or the column matrix,

$$\left\| \underline{u}_i(t) \cdot \underline{v}_j(t) \right\| , \tag{2-10}$$

where, in each case,

$$i = 1, 2, 3, \quad \text{and} \quad j = 1 \ .$$

No confusion will result if we write these, respectively, in the forms

$$\left\| \underline{v}_i (t) \cdot \underline{u}_i (t) \right\| , \tag{2-11}$$

and

$$\left\| \underline{u}_i (t) \cdot \underline{v}_i (t) \right\| , \tag{2-12}$$

or, more simply, in the forms

$$\left\| \underline{v}(t) \cdot \underline{u}_i (t) \right\| , \tag{2-13}$$

and

$$\left\| \underline{u}_i (t) \cdot \underline{v}(t) \right\| , \tag{2-14}$$

respectively.

These representations of a vector, $\underline{v}(t)$, in terms of its components with respect to the coordinate system, $\underline{u}_i (t)$, will, then, denote, respectively, the following row and column matrices:

$$\begin{pmatrix} \underline{v}(t) \cdot \underline{u}_1 (t) & \underline{v}(t) \cdot \underline{u}_2 (t) & \underline{v}(t) \cdot \underline{u}_3 (t) \end{pmatrix}, \tag{2-15}$$

and

$$\begin{pmatrix} \underline{u}_1 (t) \cdot \underline{v}(t) \\ \underline{u}_2 (t) \cdot \underline{v}(t) \\ \underline{u}_3 (t) \cdot \underline{v}(t) \end{pmatrix} . \tag{2-16}$$

The transformation which carries the representation of a vector from its components relative to one inertially oriented coordinate system, e.g., $\underline{u}_i\left(t_1\right)$, to its components relative to another inertially oriented coordinate system on the same origin, $\underline{u}_j\left(t_2\right)$, say, is symbolized by the matrix

$$\left\|\underline{u}_j\left(t_2\right) \cdot \underline{u}_i\left(t_1\right)\right\| , \qquad (2\text{-}17)$$

which can be written in the expanded form,

$$\begin{pmatrix} \underline{u}_1\left(t_2\right) \cdot \underline{u}_1\left(t_1\right) & \underline{u}_1\left(t_2\right) \cdot \underline{u}_2\left(t_1\right) & \underline{u}_1\left(t_2\right) \cdot \underline{u}_3\left(t_1\right) \\ \underline{u}_2\left(t_2\right) \cdot \underline{u}_1\left(t_1\right) & \underline{u}_2\left(t_2\right) \cdot \underline{u}_2\left(t_1\right) & \underline{u}_2\left(t_2\right) \cdot \underline{u}_3\left(t_1\right) \\ \underline{u}_3\left(t_2\right) \cdot \underline{u}_1\left(t_1\right) & \underline{u}_3\left(t_2\right) \cdot \underline{u}_2\left(t_1\right) & \underline{u}_3\left(t_2\right) \cdot \underline{u}_3\left(t_1\right) \end{pmatrix} . \qquad (2\text{-}18)$$

Thus, one has, for example,

$$\left\|\underline{u}_j\left(t_2\right) \cdot \underline{v}(t)\right\| = \left\|\underline{u}_j\left(t_2\right) \cdot \underline{u}_i\left(t_1\right)\right\| \left\|\underline{u}_i\left(t_1\right) \cdot \underline{v}(t)\right\| . \qquad (2\text{-}19)$$

We make the conventional definitions:

$$I \equiv \left\|\delta_{ij}\right\| , \qquad (2\text{-}20)$$

where $i, j = 1, 2, 3$, and δ_{ij} is Kronecker's delta, and

$$\left\|\underline{u}_i\left(t_2\right) \cdot \underline{u}_j\left(t_1\right)\right\|' \equiv \left\|\underline{u}_j\left(t_1\right) \cdot \underline{u}_i\left(t_2\right)\right\| , \qquad (2\text{-}21)$$

where $i, j = 1, 2, 3$.

We also note that such a linear transformation is orthogonal if and only if

$$\left\|\underline{u}_j\left(t_2\right) \cdot \underline{u}_i\left(t_1\right)\right\| \left\|\underline{u}_j\left(t_2\right) \cdot \underline{u}_i\left(t_1\right)\right\|' = I , \qquad (2\text{-}22)$$

and

$$\left\|\underline{u}_j(t_2)\cdot\underline{u}_i(t_1)\right\|'\left\|\underline{u}_j(t_2)\cdot\underline{u}_i(t_1)\right\| = I\ . \tag{2-23}$$

Then

$$\left\|\underline{u}_j(t_2)\cdot\underline{u}_i(t_1)\right\|^{-1} = \left\|\underline{u}_j(t_2)\cdot\underline{u}_i(t_1)\right\|'\ . \tag{2-24}$$

Consider the coordinate systems

$${}_{\oplus}\underline{u}_{i\,\oplus TG}(t)\ , \quad \text{and} \quad {}_{\oplus}\underline{u}_{i\,\oplus T(G)}(t)\ ,$$

described in connection with (2-6) and (2-7). We specify the rotation of the latter relative to the former by means of the vector

$$\underline{\omega}_{\oplus}(t)\ , \tag{2-25}$$

which is on the instantaneous axis of rotation, its direction being related to the instantaneous sense of rotation by the rule of the right-hand screw, and its magnitude being equal to the instantaneous rate of rotation.

We represent this vector by writing, for example,

$$\left\|{}_{\oplus}\underline{u}_{i\,\oplus TG}(t)\cdot\underline{\omega}_{\oplus}(t)\right\|. \tag{2-26}$$

We note also that

$$\left\|{}_{\oplus}\underline{u}_{i\,\oplus T(G)}(t)\cdot\underline{\omega}_{\oplus}(t)\right\| = \left\|{}_{\oplus}\underline{u}_{i\,\oplus TG}(t)\cdot\underline{\omega}_{\oplus}(t)\right\|\ \cdot \tag{2-27}$$

We note that, for a position vector, $\underline{r}(t)$,

$$\left\| {}_{\oplus}\underline{u}_{i\,\oplus TG}(t) \cdot \underline{r}(t) \right\| = \left\| {}_{\oplus}\underline{u}_{i\,\oplus T(G)}(t) \cdot \underline{r}(t) \right\| . \tag{2-28}$$

We denote the time derivative of the vector, $\underline{r}(t)$, when it is referred to the rotating coordinate system,

$${}_{\oplus}\underline{u}_{i\,\oplus T(G)}(t) ,$$

by writing

$$\frac{d\underline{r}_{(\oplus)}(t)}{dt} , \tag{2-29a}$$

or, simply,

$$\dot{\underline{r}}_{(\oplus)}(t) , \tag{2-29b}$$

and

$$\left\| {}_{\oplus}\underline{u}_{i\,\oplus T(G)}(t) \cdot \dot{\underline{r}}_{(\oplus)}(t) \right\| . \tag{2-30}$$

Similarly, we denote the time derivative of a vector, $\underline{r}(t)$, when it is referred to an inertially oriented coordinate system such as

$${}_{\oplus}\underline{u}_{i\,\oplus TG}(t) ,$$

by writing

$$\frac{d\underline{r}(t)}{dt} , \tag{2-31a}$$

or, simply,

$$\dot{\underline{r}}(t) \; . \tag{2-31b}$$

and

$$\left\| {}_{\oplus}\underline{u}_{i\oplus TG}(t) \cdot \dot{\underline{r}}(t) \right\| \; . \tag{2-32}$$

We have

$$\begin{aligned}\left\| {}_{\oplus}\underline{u}_{i\oplus TG}(t) \cdot \dot{\underline{r}}(t) \right\| &= \left\| {}_{\oplus}\underline{u}_{i\oplus T(G)}(t) \cdot \dot{\underline{r}}_{(\oplus)}(t) \right\| + \left\| {}_{\oplus}\underline{u}_{i\oplus T(G)}(t) \cdot \left[\underline{\omega}_{\oplus}(t) \times \underline{r}(t)\right] \right\| \\ &= \left\| {}_{\oplus}\underline{u}_{i\oplus T(G))}(t) \cdot \dot{\underline{r}}_{(\oplus)}(t) \right\| + \left\| {}_{\oplus}\underline{u}_{i\oplus TG}(t) \cdot \left[\underline{\omega}_{\oplus}(t) \times \underline{r}(t)\right] \right\| \; ,\end{aligned} \tag{2-33}$$

where the latter relation follows in view of (2-27) and (2-28).

We consider next the case of two inertially oriented coordinate systems which are oriented in the same way, i.e. whose coordinate axes are, respectively, parallel to one another, but whose origins do not coincide. In order to make the discussion specific, we select,

$$ {}_{\oplus}\underline{u}_{i\oplus T}(t) \; , $$

as one of the coordinate systems.

We represent the position vector of the center of mass of the moon, for example, in this system by writing

$$\underline{☾}(t) \; , \tag{2-34}$$

and

$$\left\| {}_{\oplus}\underline{u}_{i\,\oplus T}(t) \cdot \underline{\mathbb{C}}(t) \right\| \ . \tag{2-35}$$

We define, then, a coordinate system,

$$ {}_{\mathbb{C}}\underline{u}_{i\,\oplus T}(t) \ . \tag{2-36}$$

having its origin at the center of mass of the moon and its coordinate directions parallel, respectively, to those of

$$ {}_{\oplus}\underline{u}_{i\,\oplus T}(t) \ .$$

We denote a position vector referred to the coordinate system,

$$ {}_{\oplus}\underline{u}_{i\,\oplus T}(t) \ ,$$

by writing

$$ {}_{\oplus}\underline{r}(t) \ , \tag{2-37}$$

and

$$\left\| {}_{\oplus}\underline{u}_{i\,\oplus T}(t) \cdot {}_{\oplus}\underline{r}(t) \right\| \ . \tag{2-38}$$

We define the corresponding position vector,

$$ {}_{\mathbb{C}}\underline{r}(t) \ , \tag{2-39}$$

which is represented by

$$\left\| {}_{\mathbb{C}}\underline{u}_{i\,\oplus T}(t) \cdot {}_{\mathbb{C}}\underline{r}(t) \right\| \ , \tag{2-40}$$

by means of the relation

$$\left\| {}_{\mathbb{C}}\underline{u}_{i \oplus T}(t) \cdot {}_{\mathbb{C}}\underline{r}(t) \right\| = \left\| {}_{\oplus}\underline{u}_{i \oplus T}(t) \cdot {}_{\oplus}\underline{r}(t) \right\| - \left\| {}_{\oplus}\underline{u}_{i \oplus T}(t) \cdot \underline{\mathbb{C}}(t) \right\| \cdot \tag{2-41}$$

We represent the time derivative of, $\underline{\mathbb{C}}(t)$, by writing

$$\frac{d\underline{\mathbb{C}}(t)}{dt} \quad \text{or} \quad \dot{\underline{\mathbb{C}}}(t) \ , \tag{2-42}$$

and

$$\left\| {}_{\oplus}\underline{u}_{i \oplus T}(t) \cdot \dot{\underline{\mathbb{C}}}(t) \right\| \ . \tag{2-43}$$

We denote the time derivatives of ${}_{\oplus}\underline{r}(t)$ and ${}_{\mathbb{C}}\underline{r}(t)$, respectively, by

$${}_{\oplus}\dot{\underline{r}}(t) \quad \text{and} \quad {}_{\mathbb{C}}\dot{\underline{r}}(t) \ , \tag{2-44}$$

and represent them, respectively, by writing

$$\left\| {}_{\oplus}\underline{u}_{i \oplus T}(t) \cdot {}_{\oplus}\dot{\underline{r}}(t) \right\| \ , \tag{2-45}$$

and

$$\left\| {}_{\mathbb{C}}\underline{u}_{i \oplus T}(t) \cdot {}_{\mathbb{C}}\dot{\underline{r}}(t) \right\| \ . \tag{2-46}$$

The correction between these two vector representations is then specified by

$$\left\| {}_{\mathbb{C}}\underline{u}_{i \oplus T}(t) \cdot {}_{\mathbb{C}}\dot{\underline{r}}(t) \right\| = \left\| {}_{\oplus}\underline{u}_{i \oplus T}(t) \cdot {}_{\oplus}\dot{\underline{r}}(t) \right\| - \left\| {}_{\oplus}\underline{u}_{i \oplus T}(t) \cdot \dot{\underline{\mathbb{C}}}(t) \right\| \ . \tag{2-47}$$

When transformations, such as rotational ones, are independent of the origin and the meaning is clear, we will either omit the symbol denoting the origin or, when it facilitates the discussion, write a general index symbol, ν, denoting the origin and indicate its range, e.g.,

$$\left\| {}_{\nu}\underline{u}_{j\oplus M}\left(t_2\right) \cdot \underline{v}(t)\right\| = \left\| {}_{\nu}\underline{u}_{j\oplus M}\left(t_2\right) \cdot {}_{\nu}\underline{u}_{i\oplus M}\left(t_1\right)\right\| \left\| {}_{\nu}\underline{u}_{i\oplus M}\left(t_1\right) \cdot \underline{v}(t)\right\| , \quad (2\text{-}48)$$

where $\nu = \oplus, ☾$. When the range of ν is not indicated, this latter range will be understood.

C. Time Systems

The Goddard General Orbit Determination System employs both the ephemeris time system and the universal time system.

Definitions of these systems are contained in Reference 1. We employ these definitions and, generally, utilize the notational conventions found in that discussion. Supplements and other modifications to those conventions will be noted.

We will, for example, use the symbol,

$$T_{EJ}(t) , \quad (2\text{-}49)$$

to denote the number of Julian centuries of 36525 days, each of 86,400 ephemeris seconds, in the interval of ephemeris time from the fundamental epoch, 1900 January 0 at 12^h ephemeris time = 1900 January 0.5 E.T. = J.E.D. 241 5020.0 E.T., to the ephemeris time, t.

We will use the symbol,

$$T_{UJ}(t) , \quad (2\text{-}50)$$

to denote the number of Julian centuries of 36525 days, each of 86,400 seconds of universal time in the interval from the fundamental epoch, 1900 January 0 at

12^h universal time = 1900 January 0.5 U.T. = J.D. 241 5020.0 U.T., to the universal time, t.

The connection between E.T. and U.T. is written:

$$\Delta T = E.T. - U.T. \quad . \tag{2-51}$$

Whenever it is appropriate, we make the interpretation

$$\Delta T = E.T. - U.T.\ 2 \quad . \tag{2-52}$$

We will use the symbol,

$$T_{Et}\ (t) \ , \tag{2-53}$$

to denote the number of tropical centuries of 36524, 21988 ephemeris days, each of 86,400 ephemeris seconds, in the interval of ephemeris time from the fundamental epoch, 1900.0 = 1900 January $0^d_.814$ E.T. = J.E.D. 241 5020.314 E.T., to the ephemeris time, t.

We also write

$$T_{EJ}\left(t_2 - t_1\right) \equiv T_{EJ}\left(t_2\right) - T_{EJ}\left(t_1\right) \ , \tag{2-54}$$

$$T_{UJ}\left(t_2 - t_1\right) \equiv T_{UJ}\left(t_2\right) - T_{UJ}\left(t_1\right) \ , \tag{2-55}$$

and

$$T_{Et}\left(t_2 - t_1\right) \equiv T_{Et}\left(t_2\right) - T_{Et}\left(t_1\right) \ . \tag{2-56}$$

III. COORDINATE TRANSFORMATIONS

The next portion of the discussion is concerned with transformations.

A. Fundamental Reference Systems

A linear transformation of fundamental importance is the one relating coordinates referred to the mean equator and equinox of an epoch, t_1, to coordinates referred to the mean equator and equinox of another epoch, t_2, say.

We particularize our previous discussion by writing, for the case $t_1 \leqq t_2$,

$$\left\| \underline{u}_{jn}(t_2) \cdot \underline{v}(t) \right\| = \left\| \underline{u}_{jn}(t_2) \cdot \underline{u}_{in}(t_1) \right\| \left\| \underline{u}_{in}(t_1) \cdot \underline{v}(t) \right\| , \tag{3-1}$$

in the manner of (2-14) and (2-16) through (2-19).

We write,

$$\begin{aligned} \zeta_{0,1,2} &= \zeta_0(t_1, t_2) \\ &= \left[2304''\!.250 + 1''\!.396\, T_{Et}(t_1)\right] T_{Et}(t_2 - t_1) \\ &\quad + 0''\!.302\, T_{Et}^2(t_2 - t_1) + 0''\!.018\, T_{Et}^3(t_2 - t_1) , \end{aligned} \tag{3-2}$$

$$\begin{aligned} Z_{1,2} &= Z(t_1, t_2) \\ &= \zeta_{0,1,2} + 0''\!.791\, T_{Et}^2(t_2 - t_1) , \end{aligned} \tag{3-3}$$

and

$$\begin{aligned} \theta_{1,2} &= \theta(t_1, t_2) \\ &= \left[2004''\!.682 - 0''\!.853\, T_{Et}(t_1)\right] T_{Et}(t_2 - t_1) \\ &\quad - 0''\!.426\, T_{Et}^2(t_2 - t_1) - 0''\!.042\, T_{Et}^3(t_2 - t_1) , \end{aligned} \tag{3-4}$$

employing the notation of (2-53) and (2-56). Then, in accordance with Reference 1,

$$\underline{u}_{1n}(t_2) \cdot \underline{u}_{1n}(t_1) = \cos\zeta_{0,1,2}\cos\theta_{1,2}\cos Z_{1,2} - \sin\zeta_{0,1,2}\sin Z_{1,2}, \quad (3\text{-}5)$$

$$\underline{u}_{1n}(t_2) \cdot \underline{u}_{2n}(t_1) = -\sin\zeta_{0,1,2}\cos\theta_{1,2}\cos Z_{1,2} - \cos\zeta_{0,1,2}\sin Z_{1,2}, \quad (3\text{-}6)$$

$$\underline{u}_{1n}(t_2) \cdot \underline{u}_{3n}(t_1) = -\sin\theta_{1,2}\cos Z_{1,2}, \quad (3\text{-}7)$$

$$\underline{u}_{2n}(t_2) \cdot \underline{u}_{1n}(t_1) = \cos\zeta_{0,1,2}\cos\theta_{1,2}\sin Z_{1,2} + \sin\zeta_{0,1,2}\cos Z_{1,2}, \quad (3\text{-}8)$$

$$\underline{u}_{2n}(t_2) \cdot \underline{u}_{2n}(t_1) = -\sin\zeta_{0,1,2}\cos\theta_{1,2}\sin Z_{1,2} + \cos\zeta_{0,1,2}\cos Z_{1,2}, \quad (3\text{-}9)$$

$$\underline{u}_{2n}(t_2) \cdot \underline{u}_{3n}(t_1) = -\sin\theta_{1,2}\sin Z_{1,2}, \quad (3\text{-}10)$$

$$\underline{u}_{3n}(t_2) \cdot \underline{u}_{1n}(t_1) = \cos\zeta_{0,1,2}\sin\theta_{1,2}, \quad (3\text{-}11)$$

$$\underline{u}_{3n}(t_2) \cdot \underline{u}_{2n}(t_1) = -\sin\zeta_{0,1,2}\sin\theta_{1,2}, \quad (3\text{-}12)$$

$$\underline{u}_{3n}(t_2) \cdot \underline{u}_{3n}(t_1) = \cos\theta_{1,2}. \quad (3\text{-}13)$$

We note that the matrix,

$$\left\| \underline{u}_{jn}(t_2) \cdot \underline{u}_{in}(t_1) \right\|, \quad (3\text{-}14)$$

is orthogonal.

The transformation carrying coordinates referred to the mean equator and equinox of an epoch, t, into coordinates referred to the true equator and equinox of this epoch, is indicated as follows:

$$\left\| \underline{u}_{jT}(t) \cdot \underline{v}(t) \right\| = \left\| \underline{u}_{jT}(t) \cdot \underline{u}_{in}(t) \right\| \left\| \underline{u}_{in}(t) \cdot \underline{v}(t) \right\| , \tag{3-15}$$

in terms of (2-14) and (2-16) through (2-19). This transformation,

$$\left\| \underline{u}_{iT}(t) \cdot \underline{u}_{jn}(t) \right\| ,$$

where i, j = 1, 2, 3, is conventionally specified as follows:

$$\left\| \underline{u}_{iT}(t) \cdot \underline{u}_{jn}(t) \right\| = \begin{Vmatrix} 1 & -\Delta\psi(t)\cos\epsilon(t) & -\Delta\psi(t)\sin\epsilon(t) \\ +\Delta\psi(t)\cos\epsilon(t) & 1 & -\Delta\epsilon(t) \\ +\Delta\psi(t)\sin\epsilon(t) & \Delta\epsilon(t) & 1 \end{Vmatrix} , \tag{3-16}$$

in terms of the obliquity of the ecliptic, $\epsilon(t)$, the nutation in longitude, $\Delta\psi(t)$, and the nutation in obliquity, $\Delta\epsilon(t)$. (cf. Reference 1 and (3-53) and (3-54).) Terms of the second order are neglected in the transformation (3-16) which was just specified. These amount to no more than a part in 10^8. This transformation is, accordingly, not orthogonal, strictly speaking. It is convenient, however, to think of this transformation as being orthogonal to the first order, i.e., the orthogonality relationships are satisfied to within quantities of the second order. Thus, for example, while,

$$\left\| \underline{u}_{iT}(t) \cdot \underline{u}_{jn}(t) \right\| \left\| \underline{u}_{iT}(t) \cdot \underline{u}_{jn}(t) \right\|' \neq I , \tag{3-17}$$

where i, j = 1, 2, 3, strictly speaking, we go on to write, in addition:

$$\left\| \underline{u}_{iT}(t) \cdot \underline{u}_{jn}(t) \right\| \left\| \underline{u}_{iT}(t) \cdot \underline{u}_{jn}(t) \right\|' \doteq 8 , \tag{3-18}$$

where i, j = 1, 2, 3, where, here, the symbol

$$\stackrel{1}{=} \tag{3-19}$$

denotes the fact that the left and right hand members are equivalent to the first order. Similarly,

$$\left\| \underline{u}_{iT}(t) \cdot \underline{u}_{jn}(t) \right\|^{-1} \stackrel{1}{=} \left\| \underline{u}_{iT}(t) \cdot \underline{u}_{jn}(t) \right\|^{-1} , \tag{3-20}$$

where i, j = 1, 2, 3, etc. (cf. (2-20) through (2-24).) One can obtain

$$\left\| \underline{u}_{iT}(t_2) \cdot \underline{v}(t) \right\| , \tag{3-21}$$

from

$$\left\| \underline{u}_{iT}(t_1) \cdot \underline{v}(t) \right\| , \tag{3-22}$$

where

$$t_1 \leqq t_2 ,$$

in the following conventional manner employing (3-1) through (3-16) and (3-20):

$$\left\| \underline{u}_{jn}(t_1) \cdot \underline{v}(t) \right\| = \left\| \underline{u}_{iT}(t_1) \cdot \underline{u}_{jn}(t_1) \right\|^{-1} \left\| \underline{u}_{iT}(t_1) \cdot \underline{v}(t) \right\| , \tag{3-23}$$

$$\left\| \underline{u}_{kn}(t_2) \cdot \underline{v}(t) \right\| = \left\| \underline{u}_{kn}(t_2) \cdot \underline{u}_{jn}(t_1) \right\| \left\| \underline{u}_{jn}(t_1) \cdot \underline{v}(t) \right\| , \tag{3-24}$$

$$\left\| \underline{u}_{\ell T}(t_2) \cdot \underline{v}(t) \right\| = \left\| \underline{u}_{\ell T}(t_2) \cdot \underline{u}_{kn}(t_2) \right\| \left\| \underline{u}_{kn}(t_2) \cdot \underline{v}(t) \right\| \cdot \tag{3-25}$$

It is convenient to combine these three equations by writing

$$\left\| \underline{u}_{\ell T}(t_2) \cdot \underline{v}(t) \right\| = \left\| \underline{u}_{\ell T}(t_2) \cdot \underline{u}_{iT}(t_1) \right\| \left\| \underline{u}_{iT}(t_1) \cdot \underline{v}(t) \right\| , \tag{3-26}$$

where we have employed the product matrix

$$\left\|\underline{u}_{\ell T}(t_2)\cdot\underline{u}_{iT}(t_1)\right\| = \left\|\underline{u}_{\ell T}(t_2)\cdot\underline{u}_{kn}(t_2)\right\|$$

$$\times\left\|\underline{u}_{kn}(t_2)\cdot\underline{u}_{jn}(t_1)\right\|\times\left\|\underline{u}_{iT}(t_1)\cdot\underline{u}_{jn}(t_1)\right\|^{-1}\cdot \qquad (3\text{-}27)$$

For $t_2 < t_1$, we also have the relation (3-26) where, now,

$$\left\|\underline{u}_{\ell T}(t_2)\cdot\underline{u}_{iT}(t_1)\right\| = \left\|\underline{u}_{\ell T}(t_2)\cdot\underline{u}_{kn}(t_2)\right\|$$

$$\times\left\|\underline{u}_{jn}(t_1)\cdot\underline{u}_{kn}(t_2)\right\|^{-1}\times\left\|\underline{u}_{iT}(t_1)\cdot\underline{u}_{jn}(t_1)\right\|^{-1}\cdot \qquad (3\text{-}28)$$

It is convenient for certain purposes to employ a coordinate system,

$$_{\oplus}\underline{u}_{i\oplus TS}(t) \qquad (3\text{-}29)$$

which we define in terms of the following linear transformation,

$$\left\|{}_{\oplus}\underline{u}_{i\oplus TS}(t)\cdot{}_{\oplus}\underline{u}_{j\oplus T}(t)\right\| = \left\|\underline{u}_{iTS}(t)\cdot\underline{u}_{jT}(t)\right\|$$

$$= \left\|\begin{matrix} \cos\left[\Delta\psi(t)\cos\epsilon(t)\right] & \sin\left[\Delta\psi(t)\cos\epsilon(t)\right] & 0 \\ -\sin\left[\Delta\psi(t)\cos\epsilon(t)\right] & \cos\left[\Delta\psi(t)\cos\epsilon(t)\right] & 0 \\ 0 & 0 & 1 \end{matrix}\right\| \cdot \qquad (3\text{-}30)$$

This earth-centered coordinate system can be thought of as being referred to the earth's true equator and a reference direction which is associated with the mean equinox and which will be referred to as the space mean equinox or simply as the space equinox. Thus

$$\left\|\underline{u}_{jTS}(t_1)\cdot\underline{v}(t)\right\| = \left\|\underline{u}_{jTS}(t_1)\cdot\underline{u}_{iT}(t_1)\right\|\times\left\|\underline{u}_{iT}(t_1)\cdot\underline{v}(t)\right\|\cdot \qquad (3\text{-}31)$$

Also

$$\left\| \underline{u}_{iT}(t_1) \cdot \underline{v}(t) \right\| = \left\| \underline{u}_{jTS}(t_1) \cdot \underline{u}_{iT}(t_1) \right\|^{-1} \times \left\| \underline{u}_{jTS}(t_1) \cdot \underline{v}(t) \right\| \cdot \quad (3\text{-}32)$$

Thus, for $t_1 \leqq t_2$,

$$\left\| \underline{u}_{\ell TS}(t_2) \cdot \underline{v}(t) \right\| = \left\| \underline{u}_{\ell TS}(t_2) \cdot \underline{u}_{iTS}(t_1) \right\| \times \left\| \underline{u}_{iTS}(t_1) \cdot \underline{v}(t) \right\| , \quad (3\text{-}33)$$

where, employing (3-26) and (3-30) through (3-32), we have:

$$\left\| \underline{u}_{\ell TS}(t_2) \cdot \underline{u}_{iTS}(t_1) \right\| = \left\| \underline{u}_{\ell TS}(t_2) \cdot \underline{u}_{kT}(t_2) \right\| \left\| \underline{u}_{kT}(t_2) \cdot \underline{u}_{jT}(t_1) \right\| \times \left\| \underline{u}_{iTS}(t_1) \cdot \underline{u}_{jT}(t_1) \right\|^{-1} . \quad (3\text{-}34)$$

But

$$\left\| \underline{u}_{kTS}(t) \cdot \underline{u}_{in}(t) \right\| = \left\| \underline{u}_{kTS}(t) \cdot \underline{u}_{jT}(t) \right\| \left\| \underline{u}_{jT}(t) \cdot \underline{u}_{in}(t) \right\|$$

$$= \begin{Vmatrix} 1 & 0 & -\Delta\psi(t)\sin\epsilon(t) \\ 0 & 1 & -\Delta\epsilon(t) \\ \Delta\psi(t)\sin\epsilon(t) & \Delta\epsilon(t) & 1 \end{Vmatrix} , \quad (3\text{-}35)$$

in the sense of (3-19). This latter matrix represents the transformation from the mean equator and equinox system to the true equator and space equinox system with an accuracy equivalent to that which is associated with the conventional matrix representation, (3-16), of the transformation from the mean equator and equinox system to the true equator and equinox system. Also

$$\left\| \underline{u}_{kTS}(t) \cdot \underline{u}_{in}(t) \right\|^{-1} = \left\| \underline{u}_{jT}(t) \cdot \underline{u}_{in}(t) \right\|^{-1} \left\| \underline{u}_{kTS}(t) \cdot \underline{u}_{jT}(t) \right\|^{-1}$$

$$\doteq \begin{Vmatrix} 1 & 0 & \Delta\psi(t)\sin\epsilon(t) \\ 0 & 1 & \Delta\epsilon(t) \\ -\Delta\psi(t)\sin\epsilon(t) & -\Delta\epsilon(t) & 1 \end{Vmatrix} , \quad (3\text{-}36)$$

in the sense of (3-19). A discussion entirely analogous to the one following (3-35) applies here. Hence we may write, for $t_1 \leqq t_2$,

$$\left\| \underline{u}_{\ell TS}(t_2) \cdot \underline{u}(t) \right\| = \left\| \underline{u}_{\ell TS}(t_2) \cdot \underline{u}_{iTS}(t_1) \right\| \left\| \underline{u}_{iTS}(t_1) \cdot \underline{v}(t) \right\| , \tag{3-37}$$

where, employing now (3-1) through (3-14), (3-35) and (3-36), we have:

$$\left\| \underline{u}_{\ell TS}(t_2) \cdot \underline{u}_{iTS}(t_1) \right\| = \left\| \underline{u}_{\ell TS}(t_2) \cdot \underline{u}_{kn}(t_2) \right\| \times \left\| \underline{u}_{kn}(t_2) \cdot \underline{u}_{jn}(t_1) \right\| \times \left\| \underline{u}_{iTS}(t_1) \cdot \underline{u}_{jn}(t_1) \right\|^{-1} . \tag{3-38}$$

In similar fashion, for $t_2 < t_1$, we have the relation (3-37), where now, employing (2-24), (3-1) through (3-14), (3-35) and (3-36), we write:

$$\left\| \underline{u}_{\ell TS}(t_2) \cdot \underline{u}_{iTS}(t_1) \right\| = \left\| \underline{u}_{\ell TS}(t_2) \cdot \underline{u}_{kn}(t_2) \right\| \times \left\| \underline{u}_{jn}(t_1) \cdot \underline{u}_{kn}(t_2) \right\|^{-1} \times \left\| \underline{u}_{iTS}(t_1) \cdot \underline{u}_{jn}(t_1) \right\|^{-1} . \tag{3-39}$$

B. Earth-fixed Coordinate Systems

The position of the earth, relative to the coordinate system, $\underline{u}_{iTS}(t)$, say, can be obtained in terms of the following discussion.

Let $t_0(t)$ denote the latest epoch which is not subsequent to t, and for which the universal time is 0^h.

Let

$$R_{UGMO}\left[t_0(t)\right] = 6^h\, 38^m\, 45^s\!.836 + 86\ 40184^s\!.542\, T_{UJ}\left[t_0(t)\right] + 0^s\!.0929\, T_{UJ}^2\left[t_0(t)\right] . \tag{3-40}$$

Let

$$R_{UGM}(t) = R_{UGMO}\left[t_0(t)\right] + \left\{1.00273\ 7909265 + 0.589\, T_{UJ}\left[t_0(t)\right] \times 10^{-10}\right\} \times \left[t - t_0(t)\right] . \qquad (3\text{-}41)$$

The effect of increasing

$$T_{UJ}\left[t_0(t)\right]$$

in the last relation by

$$\frac{1}{2}\left[t - t_0(t)\right] ,$$

say, is entirely inappreciable.

In the Equation (3-41) which defines $R_{UGM}(t)$, we will, unless otherwise specified, reckon the argument, t, in terms of universal time using the measure U.T.1. If, for example, $R_{UGMO}\left[t_0(t)\right]$, and $\left[t - t_0(t)\right]$, are expressed in terms of days of time, then $R_{UGM}(t)$ can be thought of as being expressed in terms of revolutions of right ascension in Equation (3-41).

Here, one day is equivalent to 24 hours, or 1440 minutes or 86,400 seconds, and one revolution is equivalent to 2π radians or 360 degrees.

We employ, then, the following transformation to express the relationship between the coordinate systems, ${}_{\oplus}\underline{u}_{i\,\oplus TG}(t)$, and ${}_{\oplus}\underline{u}_{j\,\oplus TS}(t)$,

$$\left\|\underline{u}_{j\,TS}(t)\cdot\underline{v}(t)\right\| = \left\|\underline{u}_{j\,TS}(t)\cdot\underline{u}_{i\,TG}(t)\right\|\,\left\|\underline{u}_{i\,TG}(t)\cdot\underline{v}(t)\right\| , \qquad (3\text{-}42)$$

where i, j = 1, 2, 3, and

$$\left\| \underline{u}_{jTS}(t) \cdot \underline{u}_{iTG}(t) \right\| = \left\| \begin{matrix} \cos R_{UGM}(t) & -\sin R_{UGM}(t) & 0 \\ \sin R_{UGM}(t) & \cos R_{UGM}(t) & 0 \\ 0 & 0 & 1 \end{matrix} \right\| , \qquad (3\text{-}43)$$

and the intermediate argument is given by the relation (3-41).

The magnitude of the vector, $\underline{\omega}_{\oplus}(t)$, i.e., the earth's rate of rotation can be expressed as follows

$$\begin{aligned} \omega_{\oplus U}(t) &= |\underline{\omega}_{\oplus U}(t)| \\ &= 1.00273\ 78119.06 \end{aligned} \qquad (3\text{-}44)$$

revolutions per mean solar day of mean solar time where here we identify mean solar time with universal time (U.T.1) and the symbol U in the expressions $\underline{\omega}_{\oplus U}(t)$ and $\omega_{\oplus U}(t)$ denotes the fact that these quantities are referred to the universal time system.

We also write

$$\begin{aligned} \omega_{\oplus E}(t) &= |\underline{\omega}_{\oplus E}(t)| \\ &= \frac{1.00273\ 78119.06}{d(t)} \end{aligned} \qquad (3\text{-}45)$$

revolutions per ephemeris second, where d(t) denotes the number of ephemeris seconds in the mean solar day at the epoch t. The quantities $\underline{\omega}_{\oplus E}(t)$, and $\omega_{\oplus E}(t)$ are thus referred to the ephemeris time system. Where no confusion will result, we will denote them simply by

$$\underline{\omega}_{\oplus}(t) , \quad \text{and} \quad \omega_{\oplus}(t) , \tag{3-46}$$

respectively.

It is convenient to express the denominator in the right-hand member of (3-45) in the following way:

$$d(t) = 86,400 \left[1 + s(t) \right] . \tag{3-47}$$

The function, $s(t)$, has, at times, been of the order of

$$150 \times 10^{-10} .$$

Seasonal fluctuations in U.T. 1 were calculated in advance for the years 1956 to 1960 by means of the formula

$$\begin{aligned} &+ 0^{\mathrm{s}}.022 \sin 2\pi t - 0^{\mathrm{s}}.017 \cos 2\pi t \\ &- 0^{\mathrm{s}}.007 \sin 4\pi t + 0^{\mathrm{s}}.006 \cos 4\pi t , \end{aligned} \tag{3-48}$$

where t denotes the fraction of the year reckoned from January 1.

The variations in $s(t)$ associated with this formula are, at times, only half an order of magnitude smaller than the value cited above for $s(t)$. These quantities result in velocity corrections which are a couple of orders of magnitude smaller than a millimeter per second. Accordingly, for the moment, they are negligible.

C. An Ecliptic Reference System

We define an ecliptic coordinate system,

$$\underline{u}_{i\,C\oplus M}(t) \;, \tag{3-49}$$

as follows in terms of the coordinate system, $\underline{u}_{i\,\oplus M}(t)$, which is discussed in connection with (2-2), (2-8), and (3-1) through (3-14):

$$\underline{u}_{1C\oplus M}(t) = \underline{u}_{1\oplus M}(t) \;, \tag{3-50}$$

$$\underline{u}_{3C\oplus M}(t) = -\sin\epsilon(t)\,\underline{u}_{2\oplus M}(t) + \cos\epsilon(t)\,\underline{u}_{3\oplus M}(t) \;, \tag{3-51}$$

$$\underline{u}_{2C\oplus M}(t) = \underline{u}_{3C\oplus M}(t) \times \underline{u}_{1C\oplus M}(t) \;. \tag{3-52}$$

Here $\epsilon(t)$, the obliquity of the ecliptic, is given by,

$$\begin{aligned}\epsilon(t) = {} & 23^\circ\, 27'\, 08''\!.26 - 46''\!.845\, T_{Et}(t) \\ & - 0''\!.0059\, T_{Et}^2(t) + 0''\!.00181\, T_{Et}^3(t) \;,\end{aligned} \tag{3-53}$$

in terms of the quantity (2-53).

This quantity, $\epsilon(t)$, provides the measure of the angle between the ecliptic and the earth's mean equator at the epoch t. When it is appropriate, this quantity may be written as $\epsilon_M(t)$, and referred to as the mean obliquity. The symbol $\epsilon_T(t)$, will denote the true obliquity which is defined analogously in terms of the earth's true equator. Thus,

$$\epsilon_T(t) = \epsilon_M(t) + \Delta\epsilon(t) \;. \tag{3-54}$$

D. Lunar Coordinate Systems

A lunar oriented coordinate system, ${}_{\oplus}\underline{u}_{i☾M}(t)$, is defined in terms of the following discussion.

Let

$$☾(t) = 270°\,26'\,02''.99 + 1336^{r}\,307°\,52'\,59''.31\,T_{EJ}(t) - 4''.08\,T_{EJ}^{2}(t) + 0''.0068\,T_{EJ}^{3}(t)\,, \qquad (3\text{-}55)$$

and

$$\Omega_{☾}(t) = 259°\,10'\,59''.79 - 5^{r}\,134°\,08'\,31''.23\,T_{EJ}(t) + 7''.48\,T_{EJ}^{2}(t) + 0''.008\,T_{EJ}^{3}(t)\,, \qquad (3\text{-}56)$$

in terms of the quantity (2-49).

Here $☾(t)$ denotes the mean longitude of the moon, measured in the ecliptic from the mean equinox of epoch t to the mean ascending node of the lunar orbit, and then along the orbit, and $\Omega_{☾}(t)$ denotes the longitude of the mean ascending node of the lunar orbit on the ecliptic, measured from the mean equinox of date. The inclination of the mean lunar equator to the ecliptic will be denoted by I, or by I_M, when the latter, more detailed symbol is called for. The ascending node of the mean lunar equator on the ecliptic is at the descending node of the mean lunar orbit, i.e., at $\Omega_{☾}(t) \pm 180°$. Then, in terms of (3-49) through (3-52), and (3-56), we have:

$$\underline{u}_{1☾M}(t) = \cos\Omega_{☾}(t)\,\underline{u}_{1C\oplus M}(t) + \sin\Omega_{☾}(t)\,\underline{u}_{2C\oplus M}(t)\,, \qquad (3\text{-}57)$$

$$\underline{u}_{3\mathbb{C}M}(t) = \sin I\, \underline{u}_{3C\oplus M}(t) \times \underline{u}_{1\mathbb{C}M}(t) + \cos I\, \underline{u}_{3C\oplus M}(t) , \tag{3-58}$$

$$\underline{u}_{2\mathbb{C}M}(t) = \underline{u}_{3\mathbb{C}M}(t) \times \underline{u}_{1\mathbb{C}M}(t) . \tag{3-59}$$

The prime meridian of the moon is defined to be on the direction toward the earth when the moon is simultaneously at its mean longitude and its mean ascending node. The mean angular distance of the prime meridian of the moon from the descending node of the mean lunar equator is then given by,

$$p_{\mathbb{C}}(t) = 180^\circ + \mathbb{C}(t) - \Omega_{\mathbb{C}}(t) , \tag{3-60}$$

in terms of (3-55) and (3-56).

A coordinate system associated with the mean lunar equator and the mean position of the prime meridian of the moon can be defined in the following way in terms of (3-57) through (3-60).

$$_{\nu}\underline{u}_{1\mathbb{C}M(p)}(t) = \cos p_{\mathbb{C}}(t)\, _{\nu}\underline{u}_{1\mathbb{C}M}(t) + \sin p_{\mathbb{C}}(t)\, _{\nu}\underline{u}_{2\mathbb{C}M}(t) , \tag{3-61}$$

$$_{\nu}\underline{u}_{2\mathbb{C}M(p)}(t) = -\sin p_{\mathbb{C}}(t)\, _{\nu}\underline{u}_{1\mathbb{C}M}(t) + \cos p_{\mathbb{C}}(t)\, _{\nu}\underline{u}_{2\mathbb{C}M}(t) , \tag{3-62}$$

$$_{\nu}\underline{u}_{3\mathbb{C}M(p)}(t) = {}_{\nu}\underline{u}_{3\mathbb{C}M}(t) . \tag{3-63}$$

The inclination of the true lunar equator to the ecliptic is specified as follows:

$$I_T(t) = I_M + \rho(t) = I + \rho(t) . \tag{3-64}$$

The descending node of the true lunar equator on the ecliptic is given by

$$\Omega_{\mathbb{C}}(t) + \sigma(t) . \tag{3-65}$$

The angular distance from the descending node of the true lunar equator on the ecliptic to the true position of the prime meridian of the moon is given by

$$p_{☾}(t) + T(t) - \sigma(t) . \tag{3-66}$$

Expressions for the functions $\rho(t)$, $\sigma(t)$, and $T(t)$, are due to Hayn. (Cf. References 3 and 4.)

Coordinate systems associated with the true equator of the moon can then be defined in the following way

$$
\begin{aligned}
{}_{\nu}\underline{u}_{1☾T}(t) &= \cos\left[\Omega_{☾}(t) + \sigma(t)\right] {}_{\nu}\underline{u}_{1C☾M}(t) \\
&\quad - \sin\left[\Omega_{☾}(t) + \sigma(t)\right] {}_{\nu}\underline{u}_{2C\oplus M}(t) , \qquad (3\text{-}67)
\end{aligned}
$$

$$
\begin{aligned}
{}_{\nu}\underline{u}_{3☾T}(t) &= \sin I_T(t)\left[{}_{\nu}\underline{u}_{3C\oplus M}(t) \times {}_{\nu}\underline{u}_{1☾T}(t)\right] \\
&\quad + \cos I_T(t)\, {}_{\nu}\underline{u}_{3C\oplus M}(t) , \qquad (3\text{-}68)
\end{aligned}
$$

$${}_{\nu}\underline{u}_{2☾T}(t) = {}_{\nu}\underline{u}_{3☾T}(t) \times {}_{\nu}\underline{u}_{1☾T} , \tag{3-69}$$

in terms of (3-49) through (3-52), (3-56), (3-64) and (3-65). Then, in terms of (3-66) through (3-69), we define a coordinate system, $\underline{u}_{i☾TS}(t)$, on the basis of the true lunar equator and a reference direction referred to as the lunar space equinox:

$$
\begin{aligned}
{}_{\nu}\underline{u}_{1☾TS}(t) &= \cos\left[T(t) - \sigma(t)\right] {}_{\nu}\underline{u}_{1☾T}(t) \\
&\quad + \sin\left[T(t) - \sigma(t)\right] {}_{\nu}\underline{u}_{2☾T}(t) , \qquad (3\text{-}70)
\end{aligned}
$$

$$ {}_{\nu}\underline{u}_{2☾TS}(t) = -\sin\left[T(t)-\sigma(t)\right] {}_{\nu}\underline{u}_{1☾T}(t) + \cos\left[T(t)-\sigma(t)\right] {}_{\nu}\underline{u}_{2☾T}(t) , \tag{3-71} $$

$$ {}_{\nu}\underline{u}_{3☾TS}(t) = {}_{\nu}\underline{u}_{3☾T}(t) . \tag{3-72} $$

A coordinate system, $\underline{u}_{i☾Tp}(t)$, associated with the moon itself, i.e., with its true equator and the true position of its prime meridian can then be defined simply in the following manner in terms of (3-60) and (3-70) through (3-72):

$$ {}_{\nu}\underline{u}_{1☾Tp}(t) = \cos p_{☾}(t)\, {}_{\nu}\underline{u}_{1☾TS}(t) + \sin p_{☾}(t)\, {}_{\nu}\underline{u}_{2☾TS}(t) , \tag{3-73} $$

$$ {}_{\nu}\underline{u}_{2☾Tp}(t) = -\sin p_{☾}(t)\, {}_{\nu}\underline{u}_{1☾TS}(t) + \cos p_{☾}(t)\, {}_{\nu}\underline{u}_{2☾TS}(t) , \tag{3-74} $$

$$ {}_{\nu}\underline{u}_{3☾Tp}(t) = {}_{\nu}\underline{u}_{3☾TS}(t) . \tag{3-75} $$

A rotating coordinate system,

$$ \underline{u}_{i☾T(p)}(t) , \tag{3-76} $$

is then defined in terms of the corresponding inertially oriented coordinate system,

$$ \underline{u}_{i☾Tp}(t) , $$

in a manner which is entirely analogous to the definition of

$$ \underline{u}_{i\oplus T(G)}(t) , $$

in terms of the coordinate system

$$ \underline{u}_{i\oplus TG}(t) , $$

which was made in the discussion in connection with (2-6) and (2-7). The rotation of the coordinate system, $\underline{u}_{i\,\mathbb{C}T(p)}(t)$, with respect to the coordinate system, $\underline{u}_{I\mathbb{C}Tp}(t)$, is specified, analogously, by means of the vector,

$$\underline{\omega}_{\mathbb{C}}(t) \ , \tag{3-77}$$

which is on the instantaneous axis of rotation of the moon, its direction being related to the instantaneous sense of rotation by the rule of the right-hand screw, and its magnitude being equal to the instantaneous rate of rotation of the moon.

The magnitude of this rate of rotation can be expressed as follows:

$$\begin{aligned} \omega_{\mathbb{C}}(t) &= |\underline{\omega}_{\mathbb{C}}(t)| \\ &= \dot{\mathbb{C}}(t) - p(t) \ , \end{aligned} \tag{3-78}$$

where we derive

$$\dot{\mathbb{C}}(t)$$

from (3-55), and make use of the relation:

$$p(t) \ \square \ 50''\!.2564 + 0''\!.0222\,T_{Et}(t) \ , \tag{3-79}$$

which specifies the annual general precession in longitude in terms of the quantity (2-53).

IV. APPLICATIONS

It is convenient for many purposes involving lunar applications to employ the coordinate system,

$$ {}_{\mathbb{C}}\underline{u}_{i\,\mathbb{C}TS}(t) \ , \tag{4-1}$$

defined in (3-70) through (3-72), which we associate with an appropriately chosen epoch, t_1, say.

The procedures involved in its use can then be indicated in the following way.

A. Vectors Associated With the Earth

We transform vectors

$$ {}_{\oplus}\underline{r}(t) \ , \tag{4-2}$$

and their time derivatives,

$$ {}_{\oplus}\dot{\underline{r}}_{(\oplus)}(t) \ , \tag{4-3}$$

which are represented, respectively, in the following ways, in accordance with the discussion associated with (2-28) through (2-30), in terms of an earth-fixed, rotating coordinate system,

$$ \left\| {}_{\oplus}\underline{u}_{i\,\oplus T(G)}(t) \cdot {}_{\oplus}\underline{r}(t) \right\| \ , \tag{4-4}$$

and

$$ \left\| {}_{\oplus}\underline{u}_{i\,\oplus T(G)}(t) \cdot {}_{\oplus}\dot{\underline{r}}_{(\oplus)}(t) \right\| \ , \tag{4-5}$$

to the corresponding vectors,

$$ {}_{\mathbb{C}}\underline{r}(t) \ , \quad \text{and} \quad {}_{\mathbb{C}}\dot{\underline{r}}(t) \ , \tag{4-6} $$

respectively, which are represented as follows in accordance with the discussion associated with (2-39) through (2-41) and (2-44) through (2-47) and (3-70) through (3-72) in terms of the inertially oriented lunar coordinate system which we have selected:

$$ \left\| {}_{\mathbb{C}}\underline{u}_{i\,\mathbb{C}TS}\left(t_1\right) \cdot \ \underline{r}(t) \right\| \ , \tag{4-7} $$

and

$$ \left\| {}_{\mathbb{C}}\underline{u}_{i\,\mathbb{C}TS}\left(t_1\right) \cdot \ \dot{\underline{r}}(t) \right\| \tag{4-8} $$

We carry out this transformation by means of the following sequence of steps.

We have, from (2-28) and (2-33)

$$ \left\| {}_{\oplus}\underline{u}_{i\,\oplus TG}(t) \cdot {}_{\oplus}\underline{r}(t) \right\| = \left\| {}_{\oplus}\underline{u}_{i\,\oplus T(G)}(t) \cdot {}_{\oplus}\underline{r}(t) \right\| \ . \tag{4-9} $$

$$ \begin{aligned} \left\| {}_{\oplus}\underline{u}_{i\,\oplus TG}(t) \cdot {}_{\oplus}\dot{\underline{r}}(t) \right\| &= \left\| {}_{\oplus}\underline{u}_{i\,\oplus T(G)}(t) \cdot {}_{\oplus}\dot{\underline{r}}_{(\oplus)}(t) \right\| \\ &+ \left\| {}_{\oplus}\underline{u}_{i\,\oplus TG}(t) \cdot \left[\underline{\omega}_{\oplus}(t) \times {}_{\oplus}\underline{r}(t) \right] \right\| \ . \end{aligned} \tag{4-10} $$

We have, then, from (3-42) and (3-43)

$$ \left\| {}_{\oplus}\underline{u}_{j\oplus TS}(t) \cdot {}_{\oplus}\underline{r}(t) \right\| = \left\| {}_{\oplus}\underline{u}_{j\oplus TS}(t) \cdot {}_{\oplus}\underline{u}_{i\oplus TG}(t) \right\| \times \left\| {}_{\oplus}\underline{u}_{i\oplus TG}(t) \cdot {}_{\oplus}\underline{r}(t) \right\| \ . \tag{4-11} $$

$$ \left\| {}_{\oplus}\underline{u}_{j\oplus TS}(t) \cdot {}_{\oplus}\dot{\underline{r}}(t) \right\| = \left\| {}_{\oplus}\underline{u}_{j\oplus TS}(t) \cdot {}_{\oplus}\underline{u}_{i\oplus TG}(t) \right\| \times \left\| {}_{\oplus}\underline{u}_{i\oplus TG}(t) \cdot {}_{\oplus}\dot{\underline{r}}(t) \right\| \ . \tag{4-12} $$

We then employ the following transformation which is discussed further below in connection with the relation (4-32):

$$\left\| {}_{\oplus}\underline{u}_{j\mathbb{C}TS}\left(t_1\right) \cdot {}_{\oplus}\underline{r}(t) \right\| = \left\| {}_{\oplus}\underline{u}_{j\mathbb{C}TS}\left(t_1\right) \cdot {}_{\oplus}\underline{u}_{i\oplus TS}(t) \right\| \times \left\| {}_{\oplus}\underline{u}_{i\oplus TS}(t) \cdot {}_{\oplus}\underline{r}(t) \right\| \cdot \quad (4\text{-}13)$$

$$\left\| {}_{\oplus}\underline{u}_{j\mathbb{C}TS}\left(t_1\right) \cdot {}_{\oplus}\underline{\dot{r}}(t) \right\| = \left\| {}_{\oplus}\underline{r}_{j\mathbb{C}TS}\left(t_1\right) \cdot {}_{\oplus}\underline{u}_{i\oplus TS}(t) \right\| \times \left\| {}_{\oplus}\underline{u}_{i\oplus TS}(t) \cdot {}_{\oplus}\underline{\dot{r}}(t) \right\| \cdot \quad (4\text{-}14)$$

We have, then, from (2-41):

$$\left\| {}_{\mathbb{C}}\underline{u}_{i\mathbb{C}TS}\left(t_1\right) \cdot {}_{\mathbb{C}}\underline{r}(t) \right\| = \left\| {}_{\oplus}\underline{u}_{i\mathbb{C}TS}\left(t_1\right) \cdot {}_{\oplus}\underline{r}(t) \right\| - \left\| {}_{\oplus}\underline{u}_{i\mathbb{C}TS}\left(t_1\right) \cdot \underline{\mathbb{C}}(t) \right\| \cdot \quad (4\text{-}15)$$

and, from (2-47),

$$\left\| {}_{\mathbb{C}}\underline{u}_{i\mathbb{C}TS}\left(t_1\right) \cdot {}_{\mathbb{C}}\underline{\dot{r}}(t) \right\| = \left\| {}_{\oplus}\underline{u}_{i\mathbb{C}TS}\left(t_1\right) \cdot {}_{\oplus}\underline{\dot{r}}(t) \right\| - \left\| {}_{\oplus}\underline{u}_{i\mathbb{C}TS}\left(t_1\right) \cdot \underline{\dot{\mathbb{C}}}(t) \right\| \cdot \quad (4\text{-}16)$$

B. Vectors Associated With the Moon

We then transform vectors

$${}_{\mathbb{C}}\underline{r}(t) \ , \qquad (4\text{-}17)$$

and their time derivatives,

$${}_{\mathbb{C}}\underline{\dot{r}}_{(\mathbb{C})}(t) \ , \qquad (4\text{-}18)$$

which are represented, respectively, in the following ways, in accordance with the discussion associated with (2-39), (2-44) and (3-76), in terms of a lunar-fixed, rotating coordinate system,

$$\left\| {}_{\mathbb{C}}\underline{u}_{i\mathbb{C}T(P)}(t) \cdot {}_{\mathbb{C}}\underline{r}(t) \right\| \ , \qquad (4\text{-}19)$$

and

$$\left\| {}_{\mathbb{C}}\underline{u}_{i\,\mathbb{C}T(P)}(t) \cdot {}_{\mathbb{C}}\dot{\underline{r}}_{(\mathbb{C})}(t) \right\| , \tag{4-20}$$

to the corresponding vectors,

$${}_{\mathbb{C}}\underline{r}(t) , \quad \text{and} \quad {}_{\mathbb{C}}\dot{\underline{r}}(t) , \tag{4-21}$$

respectively, which are represented as follows, in accordance with the discussion associated with (3-70) through (3-72), in terms of our inertially oriented lunar coordinate system:

$$\left\| {}_{\mathbb{C}}\underline{u}_{i\,\mathbb{C}TS}(t_1) \cdot {}_{\mathbb{C}}\underline{r}(t) \right\| , \tag{4-22}$$

and

$$\left\| {}_{\mathbb{C}}\underline{u}_{i\,\mathbb{C}TS}(t_1) \cdot {}_{\mathbb{C}}\dot{\underline{r}}(t) \right\| . \tag{4-23}$$

We carry out this transformation by means of the following sequence of steps.

We have, from (2-28), (2-33), and (3-73) through (3-76),

$$\left\| {}_{\mathbb{C}}\underline{u}_{i\,\mathbb{C}Tp}(t) \cdot {}_{\mathbb{C}}\underline{r}(t) \right\| = \left\| {}_{\mathbb{C}}\underline{u}_{i\,\mathbb{C}T(p)}(t) \cdot {}_{\mathbb{C}}\underline{r}(t) \right\| . \tag{4-24}$$

$$\left\| {}_{\mathbb{C}}\underline{u}_{i\mathbb{C}Tp}(t) \cdot {}_{\mathbb{C}}\dot{\underline{r}}(t) \right\| = \left\| {}_{\mathbb{C}}\underline{u}_{i\mathbb{C}T(p)}(t) \cdot {}_{\mathbb{C}}\dot{\underline{r}}_{(\mathbb{C})}(t) \right\| + \left\| {}_{\mathbb{C}}\underline{u}_{i\mathbb{C}Tp}(t) \cdot \left[\underline{\omega}_{\mathbb{C}}(t) \times {}_{\mathbb{C}}\underline{r}(t) \right] \right\| . \tag{4-25}$$

We have, from (3-73) through (3-75),

$$\left\| {}_{\mathbb{C}}\underline{u}_{j\,\mathbb{C}TS}(t) \cdot {}_{\mathbb{C}}\underline{r}(t) \right\| = \left\| {}_{\mathbb{C}}\underline{u}_{j\,\mathbb{C}TS}(t) \cdot {}_{\mathbb{C}}\underline{u}_{i\,\mathbb{C}Tp}(t) \right\| \times \left\| {}_{\mathbb{C}}\underline{u}_{i\,\mathbb{C}Tp}(t) \cdot {}_{\mathbb{C}}\underline{r}(t) \right\| . \tag{4-26}$$

$$\left\| {}_{\mathbb{C}}\underline{u}_{j\,\mathbb{C}TS}(t) \cdot {}_{\mathbb{C}}\dot{\underline{r}}(t) \right\| = \left\| {}_{\mathbb{C}}\underline{u}_{j\,\mathbb{C}TS}(t) \cdot {}_{\mathbb{C}}\underline{u}_{i\,\mathbb{C}Tp}(t) \right\| \times \left\| {}_{\mathbb{C}}\underline{u}_{i\,\mathbb{C}Tp}(t) \cdot {}_{\mathbb{C}}\dot{\underline{r}}(t) \right\| . \tag{4-27}$$

We then employ the following transformation which is discussed further below in connection with the relation (4-34):

$$\left\| {}_{\mathbb{C}}\underline{u}_{j\mathbb{C}TS}\,(t_1) \cdot {}_{\mathbb{C}}\underline{r}(t) \right\| = \left\| {}_{\mathbb{C}}\underline{u}_{j\mathbb{C}TS}\,(t_1) \cdot {}_{\mathbb{C}}\underline{u}_{i\mathbb{C}TS}\,(t) \right\| \times \left\| {}_{\mathbb{C}}\underline{u}_{i\mathbb{C}TS}\,(t) \cdot {}_{\mathbb{C}}\underline{r}(t) \right\| \cdot \tag{4-28}$$

and

$$\left\| {}_{\mathbb{C}}\underline{u}_{j\mathbb{C}TS}\,(t_1) \cdot {}_{\mathbb{C}}\dot{\underline{r}}(t) \right\| = \left\| {}_{\mathbb{C}}\underline{u}_{j\mathbb{C}TS}\,(t_1) \cdot {}_{\mathbb{C}}\underline{u}_{i\mathbb{C}TS}\,(t) \right\| \times \left\| {}_{\mathbb{C}}\underline{u}_{i\mathbb{C}TS}\,(t) \cdot {}_{\mathbb{C}}\dot{\underline{r}}(t) \right\| \cdot \tag{4-29}$$

It is convenient, in certain cases, to organize the calculations in the following way.

We consider the following product matrix which is formed on the basis of (3-49) through (3-52) and (3-67) through (3-72):

$$\begin{aligned} \left\| \underline{u}_{\ell\mathbb{C}TS}\,(t) \cdot \underline{u}_{i\oplus M}\,(t) \right\| &= \left\| \underline{u}_{\ell\mathbb{C}TS}\,(t) \cdot \underline{u}_{k\mathbb{C}T}\,(t) \right\| \\ &\quad \times \left\| \underline{u}_{k\mathbb{C}T}\,(t) \cdot \underline{u}_{jC\oplus M}\,(t) \right\| \times \left\| \underline{u}_{jC\oplus M}\,(t) \cdot \underline{u}_{i\oplus M}\,(t) \right\| \cdot \end{aligned} \tag{4-30}$$

We note that, in view of the orthogonality relationships which are satisfied,

$$\begin{aligned} \left\| \underline{u}_{\ell\mathbb{C}TS}(t) \cdot \underline{u}_{i\oplus M}(t) \right\|^{-1} &= \left\| \underline{u}_{jC\oplus M}(t) \cdot \underline{u}_{i\oplus M}(t) \right\|^{-1} \times \left\| \underline{u}_{k\mathbb{C}T}(t) \cdot \underline{u}_{jC\oplus M}(t) \right\|^{-1} \times \left\| \underline{u}_{\ell\mathbb{C}TS}(t) \cdot \underline{u}_{k\mathbb{C}T}(t) \right\|^{-1} \\ &= \left\| \underline{u}_{jC\oplus M}\,(t) \cdot \underline{u}_{i\oplus M}\,(t) \right\|^{1} \times \left\| \underline{u}_{k\mathbb{C}T}\,(t) \cdot \underline{u}_{jC\oplus M}\,(t) \right\|^{1} \times \left\| \underline{u}_{\ell\mathbb{C}TS}\,(t) \cdot \underline{u}_{k\mathbb{C}T}\,(t) \right\|^{1} \\ &= \left\| \underline{u}_{\ell\mathbb{C}TS}\,(t) \cdot \underline{u}_{i\oplus M}\,(t) \right\|^{1} \cdot \end{aligned} \tag{4-31}$$

We then form the product matrix

$$\begin{aligned} \left\| \underline{u}_{\ell\mathbb{C}TS}\,(t_1) \cdot \underline{u}_{i\oplus TS}\,(t) \right\| &= \left\| \underline{u}_{\ell\mathbb{C}TS}\,(t_1) \cdot \underline{u}_{k\oplus M}\,(t_1) \right\| \\ &\quad \times \left\| \underline{u}_{k\oplus M}\,(t_1) \cdot \underline{u}_{j\oplus M}\,(t) \right\| \times \left\| \underline{u}_{i\oplus TS}\,(t) \cdot \underline{u}_{j\oplus M}\,(t) \right\|^{-1} , \end{aligned} \tag{4-32}$$

on the basis of (3-1) through (3-14), (3-36) and (4-30). We also write

$$\left\|\underline{u}_{\ell \mathbb{C} TS}(t_1) \cdot \underline{u}_{i \oplus TS}(t)\right\| \doteq \left\|\underline{u}_{\ell \mathbb{C} TS}(t_1) \cdot \underline{u}_{k \oplus M}(t_1)\right\|$$

$$\times \left\|\underline{u}_{k \oplus M}(t_1) \cdot \underline{u}_{j \oplus M}(t)\right\| \times \left\|\underline{u}_{i \oplus TS}(t) \cdot \underline{u}_{j \oplus M}(t)\right\|^{1}, \quad (4\text{-}33)$$

on the basis of the discussion associated with (3-36). We also form the product matrix:

$$\left\|\underline{u}_{\ell \mathbb{C} TS}(t_1) \cdot \underline{u}_{i \mathbb{C} TS}(t)\right\| = \left\|\underline{u}_{\ell \mathbb{C} TS}(t_1) \cdot \underline{u}_{k \oplus M}(t_1)\right\|$$

$$\times \left\|\underline{u}_{k \oplus M}(t_1) \cdot \underline{u}_{j \oplus M}(t)\right\| \times \left\|\underline{u}_{i \mathbb{C} TS}(t) \cdot \underline{u}_{j \oplus M}(t)\right\|^{-1}, \quad (4\text{-}34)$$

on the basis of (3-1) through (3-14), (4-30) and (4-31). We note also that

$$\left\|\underline{u}_{\ell \mathbb{C} TS}(t_1) \cdot \underline{u}_{i \mathbb{C} TS}(t)\right\| = \left\|\underline{u}_{\ell \mathbb{C} TS}(t_1) \cdot \underline{u}_{k \oplus M}(t_1)\right\|$$

$$\times \left\|\underline{u}_{k \oplus M}(t_1) \cdot \underline{u}_{j \oplus M}(t)\right\| \times \left\|\underline{u}_{i \mathbb{C} TS}(t) \cdot \underline{u}_{j \oplus M}(t)\right\|^{1}, \quad (4\text{-}35)$$

on the basis of (4-31). We note again that, in view of the various orthogonality relationships which are satisfied, we have

$$\left\|\underline{u}_{\ell \mathbb{C} TS}(t_1) \cdot \underline{u}_{i \oplus TS}(t)\right\|^{-1} = \left\|\underline{u}_{i \oplus TS}(t) \cdot \underline{u}_{j \oplus M}(t)\right\|$$

$$\times \left\|\underline{u}_{k \oplus M}(t_1) \cdot \underline{u}_{j \oplus M}(t)\right\|^{-1} \times \left\|\underline{u}_{\ell \mathbb{C} TS}(t_1) \cdot \underline{u}_{k \oplus M}(t_1)\right\|^{-1}, \quad (4\text{-}36)$$

from (4-32), and

$$\left\|\underline{u}_{\ell \mathbb{C} TS}(t_1) \cdot \underline{u}_{i \oplus TS}(t)\right\|^{-1} = \left\|\underline{u}_{i \oplus TS}(t) \cdot \underline{u}_{j \oplus M}(t)\right\|$$

$$\times \left\|\underline{u}_{k \oplus M}(t_1) \cdot \underline{u}_{j \oplus M}(t)\right\|^{1} \times \left\|\underline{u}_{\ell \mathbb{C} TS}(t_1) \cdot \underline{u}_{k \oplus M}(t_1)\right\|^{-1}$$

$$= \left\| \underline{u}_{i\oplus TS}(t) \cdot \underline{u}_{j\oplus M}(t) \right\|$$

$$\times \left\| \underline{u}_{k\oplus M}(t_1) \cdot \underline{u}_{j\oplus M}(t) \right\|^{1} \times \left\| \underline{u}_{\ell \copyright TS}(t_1) \cdot \underline{u}_{k\oplus M}(t_1) \right\|^{1} , \qquad (4\text{-}37)$$

on the basis of (3-14) and (4-31), respectively. Hence,

$$\left\| \underline{u}_{\ell \copyright TS}(t_1) \cdot \underline{u}_{i\oplus TS}(t) \right\|^{-1} = \left\| \underline{u}_{\ell \copyright TS}(t_1) \cdot \underline{u}_{i\oplus TS}(t) \right\|^{1} , \qquad (4\text{-}38)$$

from (4-36) and (4-37). We also have, similarly,

$$\left\| \underline{u}_{\ell \copyright TS}(t_1) \cdot \underline{u}_{i \copyright TS}(t) \right\|^{-1} = \left\| \underline{u}_{\ell \copyright TS}(t) \cdot \underline{u}_{j\oplus M}(t) \right\|$$

$$\times \left\| \underline{u}_{k\oplus M}(t_1) \cdot \underline{u}_{j\oplus M}(t) \right\|^{-1} \times \left\| \underline{u}_{\ell \copyright TS}(t_1) \cdot \underline{u}_{k\oplus M}(t_1) \right\|^{-1}$$

$$= \left\| \underline{u}_{\ell \copyright TS}(t) \cdot \underline{u}_{j\oplus M}(t) \right\|$$

$$\times \left\| \underline{u}_{k\oplus M}(t_1) \cdot \underline{u}_{j\oplus M}(t) \right\|^{1} \times \left\| \underline{u}_{\ell \copyright TS}(t_1) \cdot \underline{u}_{k\oplus M}(t_1) \right\|^{1} , \qquad (4\text{-}39)$$

from (3-14) and (4-31), and, hence,

$$\left\| \underline{u}_{\ell \copyright TS}(t_1) \cdot \underline{u}_{i \copyright TS}(t) \right\|^{-1} = \left\| \underline{u}_{\ell \copyright TS}(t_1) \cdot \underline{u}_{i \copyright TS}(t) \right\|^{1} , \qquad (4\text{-}40)$$

from (4-39). We also utilize the product matrix

$$\left\| \underline{u}_{k \copyright TS}(t_1) \cdot \underline{u}_{i\oplus M}(t_{.5}) \right\| = \left\| \underline{u}_{k \copyright TS}(t_1) \cdot \underline{u}_{j\oplus M}(t_1) \right\| \times \left\| \underline{u}_{j\oplus M}(t_1) \cdot \underline{u}_{i\oplus M}(t_{.5}) \right\| , \qquad (4\text{-}41)$$

which is written on the basis of the relations (3-1) through (3-14), and (4-30):

C. Combination and Representation of Vectors and Transformations

In certain cases it is convenient to form and store representations of the matrices

$$\left\| \underline{u}_{\ell \mathbb{C} TS}(t_0) \cdot \underline{u}_{k \oplus TS}(t_i) \right\| , \tag{4-42}$$

for $i = 1, 2, \cdots, n$, and

$$\left\| \underline{u}_{\ell \mathbb{C} TS}(t_0) \cdot \underline{u}_{k \mathbb{C} TS}(t_j) \right\| , \tag{4-43}$$

for $j = 1, 2, \cdots, m$, which were specified in the relations (4-32) and (4-34), respectively.

Here, t_0 denotes the epoch of the reference coordinate system and, typically,

$$t_i = t_0 + i(\Delta t_{\oplus \mathbb{C}}) , \tag{4-44}$$

where $i = \pm 1, \pm 2, \cdots, \pm p$, and

$$t_j = t_0 + j(\Delta t_{\mathbb{C}}) , \tag{4-45}$$

where $j = \pm 1, \pm 2, \cdots, \pm q$, and $\Delta t_{\mathbb{C}}$ denotes a time interval associated with a polynomial representation of the precession and nutation of the moon, and $\Delta t_{\oplus \mathbb{C}}$ denotes a time interval associated with a polynomial representation of the precessions and nutations of the earth and the moon.

We employ, also, the transformation

$$\left\| {}_{\oplus}\underline{u}_{j \mathbb{C} TS}(t_c) \cdot \underline{\mathbb{C}}(t) \right\| = \left\| {}_{\oplus}\underline{u}_{j \mathbb{C} TS}(t_c) \cdot {}_{\oplus}\underline{u}_{i \oplus M}(t_{.5}) \right\| \times \left\| {}_{\oplus}\underline{u}_{i \oplus M}(t_{.5}) \cdot \underline{\mathbb{C}}(t) \right\| , \tag{4-46}$$

which utilizes the matrix specified in the relation (4-41). In certain cases it is convenient to form and store representations of the matrices

$$\left\| {}_{\oplus}\underline{u}_{j\,\mathbb{C}TS}\,(t_c) \cdot \underline{\mathbb{C}}(t_i) \right\| , \tag{4-47}$$

for $i = 1, 2, \cdots, n$, which are obtained by means of the relation (4-46).

Here, again, t_0 denotes the epoch of the coordinate system, and typically,

$$t_i = t_0 + i\left({}_{\mathbb{C}}\Delta t\right) , \tag{4-48}$$

where $i = 1, 2, \cdots, m$, and ${}_{\mathbb{C}}\Delta t$ denotes a time interval associated with a polynomial representation of the position and velocity of the moon.

V. REFERENCES

1. Explanatory Supplement to the Astronomical Ephemersis and the American Ephemeris and Nautical Almanac, Prepared Jointly by the United Kingdom and the United States of America, London, 1961.

2. Brown, E. W., Tables of the Motion of the Moon; New Haven, 1919.

3. Hayn, F., Astr. Nach., 199, 261, 1914.

4. Hayn, F., Astr. Nach., 211, 311, 1920.

January, 1968